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Forchheimer law derived by homogenization of gas flow in turbomachines☆

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Abstract

A general formulation of the homogenization problem of compressible fluid flow through a periodic porous material in turbomachines is presented here. This formulation is able to derive a Forchheimer law with a mean velocity dependent permeability as equivalent macroscopic behavior. To specify this permeability, additional flow problems are defined on the unit cell and solved by a mixed stabilized finite element discretization. The application of the Galerkin least-square (GLS) method requires the introduction of two stabilization terms with appropriate parameters. The mixed finite element discretization of these unit cell problems is finally outlined.

MSC: 76M50; 76M10; 76S07; 74A40

Keywords: Homogenization; Porous media; Mixed stabilized finite elements

1. Problem description

Efficient cooling of blades and combustion chambers in modern gas turbines is required to improve the performance of power plants. Classical film cooling can be improved by the application of transpiration cooling technology in combination with open-cell materials. At the moment, it is not possible to evaluate numerically the interaction between hot and cooling gases locally in each cooling channel of a gas turbine blade or a combustion chamber lining. Therefore, a multiscale approach based on the homogenization technique with asymptotic expansions is adopted here to calculate effective equivalent thermophysical properties. As the homogenization of the advective heat transfer problem has been performed in a previous study [9], this paper focuses on the homogenization of the compressible fluid flow through porous materials.

In the homogenization procedure applied to the fluid flow through a porous media, Stokes flow ($Re \ll 1$) is usually assumed on the unit cell [5]. This assumption leads to the classical Darcy law as equivalent macroscopic law. However, in cooling channels of gas turbine components nonlinear and locally turbulent flow conditions exist [1]. The derivation of an equivalent Forchheimer law with a velocity dependent permeability is therefore required.

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For incompressible nonlinear fluid flow only few papers [7,3] suggest nonlinear corrections to Darcy's law. For compressible nonlinear gas flow, these formulations constitute a further approximation as they neglect the dependence of the gas density with the micro/macro scale parameter ε. Although Ene and Sanchez [5] mentioned already this gas flow property, no specific multiscale model exists in the literature, which for a nonlinear compressible gas flow specifies the appropriate correction terms to Darcy's law. Therefore, a new derivation of a generalized Forchheimer law with velocity dependent permeability will be presented here. Note that for this law, Park [11] has recently proved the existence and uniqueness of a mixed finite element approximation.

This paper is organized as follows: at first, the standard setup of the homogenization technique is applied to the steady state Navier–Stokes equations of a compressible gas flow through a porous medium. Based on the flow conditions in gas turbines, a new parameter set is derived in Section 2.2 to express the dependence of the velocity, viscosity and density with the scale parameter ε. This parameter set leads to define a microscopic flow problem on the unit cell. A weak variational formulation and a split of the velocity and pressure terms are then used in Section 2.3 to solve these problems and to deduce the desired generalized Forchheimer law. In order to predict numerically the Forchheimer permeability, a stabilized mixed finite element discretization based on the GLS method is presented in Section 3.

2. Homogenization of the gas flow through a porous material

2.1. Gas flow through a heterogeneous porous medium

The cooling channels (see Fig. 1) are filled with a compressible gas whose density is small but varies mainly with the temperature due to the low pressure drop which was evaluated in the 3-D conjugate heat and fluid flow analyses of the considered multilayer plate [1]. Thus, by adopting the Boussinesq approximation [5], following state equation is adopted for the cooling gas:

$$\rho^h = \rho_0(1 - \beta T^h) \tag{1}$$

where β is the thermal expansion coefficient of the fluid: ρ_0 the density at the reference temperature.

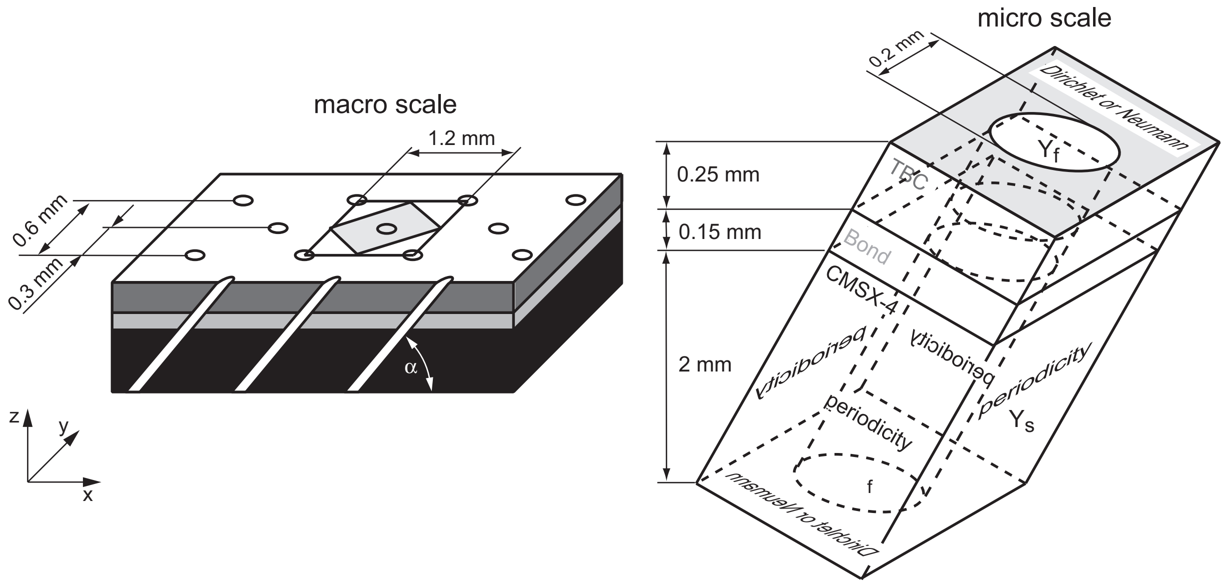

Fig. 1. Definition of a unit cell for the transpiration cooled multi-layer plate.

The steady state of the fluid flow (v^h, p^h) in the heterogeneous media Ω^h is governed by following Navier–Stokes equations:

$$\text{(a) } \textit{Momentum}: \ \rho^h v_k^h \frac{\mathrm{d}v_i^h}{\mathrm{d}z_k} = -\frac{\mathrm{d}p^h}{\mathrm{d}z_i} + \mu^h \frac{\mathrm{d}^2 v_i^h}{\mathrm{d}z_k^2} + \rho g_i, \tag{2}$$

$$\text{(b) } \textit{Continuity}: \ \frac{\mathrm{d}(\rho^h v_i)}{\mathrm{d}z_i} = 0 \tag{3}$$

and on the fluid–solid interface, Γ_{fs}^h, by the no-slip BC: $\mathbf{v} = 0$ (4)

2.2. Asymptotic expansion for flow through a porous medium

1. The homogenization technique is based on the assumption that the heterogeneous open-cell material is equivalent to a material built on the microscopic scale by periodic repetitions of a unit cell (see Fig. 1). The heterogeneous material Ω^h has a periodic microstructure where each cell on the macroscale, εY, is homothetic to the unit cell Y on the microscale with a ratio $\varepsilon \ll 1$. The unit cell Y is formed by two parts, Y_{f} and Y_{s}, representing, respectively, the fluid and solid part of the microstructure of the material.

The multiscale approach starts with a formal asymptotic expansion of the periodic unknown physical fields—velocity, pressure and temperature—on the unit cell Y with respect to the aspect ratio ε, as outlined in [12]

$$\mathbf{v}^h(\mathbf{z}) = \mathbf{v}(\mathbf{x}, \mathbf{y}) = \varepsilon^n \mathbf{v}^0(\mathbf{x}, \mathbf{y}) + \varepsilon^{n+1}\mathbf{v}^1(\mathbf{x}, \mathbf{y}) + \mathcal{O}(\varepsilon^{n+2}), \tag{5}$$

$$p^h(\mathbf{z}) = p(\mathbf{x}, \mathbf{y}) = p^0(\mathbf{x}, \mathbf{y}) + \varepsilon p^1(\mathbf{x}, \mathbf{y}) + \mathcal{O}(\varepsilon^2), \tag{6}$$

where $\mathbf{x}$ is the macroscopic variable, which varies slowly from unit cell to unit cell; $\mathbf{y} = \mathbf{x}/\varepsilon$ the periodic, microscopic variable, which describes the strong field variations within each unit cell; n a real number to be determined.

2. In a general multiscale approach, the density and viscosity depend on the scale parameter ε. Thus, we have

$$\mu^h = \varepsilon^m \mu_0 \quad \text{and} \quad \rho^h = \varepsilon^r \rho_0(1 - \beta T^h), \tag{7}$$

where m, r are real numbers to be determined.

Introducing the temperature solution of the thermal homogenization problem [9]

$$T^h(\mathbf{z}) = T^0(\mathbf{x}) - \varepsilon\chi^k(\mathbf{y})\nabla_{x_k} T^0(\mathbf{x}), \tag{8}$$

where $\chi^k (k = 1, 2, 3)$ are microscopic, periodic displacement fields in the gas state (7), leads to

$$\rho^h = \varepsilon^r \rho^*(\mathbf{x}) - \varepsilon^{r+1}\beta\chi^k(\mathbf{y})\nabla_{x_k} T^0(\mathbf{x}) + \mathcal{O}(\varepsilon^{r+2}) \quad \text{with } \rho^*(\mathbf{x}) = \rho_0[1 - \beta T^0(\mathbf{x})]. \tag{9}$$

The serial developments of the velocity and pressure fields (5)–(6) and the differential operator: $\mathrm{D}/\mathrm{D}z_i = \nabla_{x_i} + \varepsilon^{-1}\nabla_{y_i}$ are next introduced in the steady state Navier–Stokes equations (2)–(3) and in the no-slip boundary condition (4) leading to following system of differential equations:

(a) *Momentum*

$$\begin{aligned}
&\rho^*\varepsilon^{2n+r-1}(v_k^0 \cdot \nabla_{y_k} v_i^0) + \varepsilon^{2n+r}\Big\{\rho^*[v_k^0 \cdot (\nabla_{x_k} v_i^0 + \nabla_{y_k} v_i^1) + v_k^1 \cdot \nabla_{y_k} v_i^0] \\
&\quad + \rho_0\beta\chi^k\nabla_{x_k} T^0(v_k^0 \cdot \nabla_{y_k} v_i^0)\Big\} + \rho_0\varepsilon^{2n+r+1}(\ldots) = -\varepsilon^{-1}\nabla_{y_i} p^0 - \varepsilon^0(\nabla_{x_i} p^0 + \nabla_{y_i} p^1) - \varepsilon(\ldots) \\
&\quad + \mu_0\varepsilon^{n+m-2}\Delta_y v_i^0 + \varepsilon^{n+m-1}(2\nabla_{x_k} \cdot \nabla_{y_k} v_i^0 + \Delta_y v_i^1) \\
&\quad + \mu_0\varepsilon^{n+m}(\Delta_x v_i^0 + 2\nabla_{x_k} \cdot \nabla_{y_k} v_i^1) + \cdots + \rho^*\varepsilon^r g_i + \rho_0\varepsilon^{r+1}\beta g_i\chi^k\nabla_{x_k} T^0.
\end{aligned} \tag{10}$$

(b) *Continuity*:

$$(\nabla_{x_i} + \varepsilon^{-1}\nabla_{y_i})(\rho^*\varepsilon^r + \rho_0\varepsilon^{r+1}\beta\chi^k\nabla_{x_k} T^0)(\varepsilon^n v^0 + \varepsilon^{n+1} v^1 + \cdots) = 0. \tag{11}$$

(c) *Boundary condition on Γ_{fs}*:

$$\varepsilon^n v_i^0(\mathbf{x},\mathbf{y}) + \varepsilon^{n+1} v_i^1(\mathbf{x},\mathbf{y}) + \cdots = 0. \tag{12}$$

3. Before collecting the terms with the same power of ε in the system (10)–(12), the values of the parameters m, n and r must be defined. The macroscopic pressure gradient $\nabla_{x_i} p^0$ is there of order zero, while the first term of the microscopic viscous and inertial forces are of order $m+n-2$ and $2n+r-1$, respectively. Depending on the considered flow problem, different assumptions are realized:

- *Viscous flow*: in the classical formulation of Sanchez–Palencia [12], the inertial forces are neglected in the momentum equation of order zero. Only viscous forces balance the macroscopic pressure gradient leading thus to $m+n-2=0$ and to following parameters definition: $\mathbf{m}=\mathbf{0}$, $\mathbf{n}=\mathbf{2}$ and $\mathbf{r}=\mathbf{0}$. Physically, this implies that the microscopic fluid flow is in the laminar Darcy regime and governed by the Stokes equations;
- *Incompressible nonlinear flow*: in order to model nonlinear effects, Giorgi [7] and Chen et al. [3] assume that the inertial and viscous forces contribute in the same manner to balance the macroscopic pressure gradient. The order of the inertial forces must also be equal to zero: $2n+r-1=0$. Moreover, Giorgi [7] and Chen et al. [3] adopt in their formulation: $\mathbf{m}=\frac{\mathbf{3}}{\mathbf{2}}$ and $\mathbf{n}=\frac{\mathbf{1}}{\mathbf{2}}$, which implies that $\mathbf{r}=\mathbf{0}$. This choice constitutes for compressible gas flows a further approximation because it neglects the dependence of the gas density with the scale parameter ε;
- *Compressible nonlinear flow*: already Ene and Sanchez–Palencia [5] mentioned that gas flow has a small density, which depends from ε and implies: $\mathbf{r}>\mathbf{0}$. Based on this fact, we define following procedure:
  - the viscous and inertial forces continue to balance both the macroscopic pressure gradient. As they play a similar role, the conditions $m+n-2=0$ and $2n+r-1=0$ are always valid;
  - among the allowed parameter sets, we adopt: $\mathbf{n}=\frac{\mathbf{1}}{\mathbf{4}}$; $\mathbf{m}=\frac{\mathbf{7}}{\mathbf{4}}$ and $\mathbf{r}=\frac{\mathbf{1}}{\mathbf{2}}$. This specific choice is justified by flow conditions prevailing in gas turbine components and is better than the set: $n=\frac{3}{8}$; $m=\frac{13}{8}$; $r=\frac{1}{4}$. Indeed, the gas velocity reaches 10 m/s in the cooling channels of a combustion chamber. Its density at 450 °C is 0.479 kg/m^3 and its viscosity is given by 7.19×10^{-5} m^2/s. As the viscosity is small, the choice $m=\frac{7}{4}$ is better than $m=\frac{13}{8}$. Otherwise, the nonlinear velocity decreases only slowly with ε before it becomes pure laminar. Therefore, the choice $n=\frac{1}{4}$ is better than $n=\frac{3}{8}$.

4. With this parameter choice, the ε^{-1} term of the momentum (10) becomes

$$-\nabla_{y_i} p^0(\mathbf{x},\mathbf{y}) = 0 \to p^0 = p^0(\mathbf{x}). \tag{13}$$

Thus p^0 is independent of the periodic variable y and all terms involving $\nabla_{y_i} p^0(\mathbf{x},\mathbf{y})$ vanish. Then, we take the ε^0 term of the momentum (10), the $\varepsilon^{-1/4}$ term of the continuity equation (11) and the $\varepsilon^{1/4}$ term of the boundary condition (12). This extracted set of equations specifies following microscopic problem on the unit cell:

$$\rho^*(v_k^0\cdot\nabla_{y_k} v_i^0) = -\nabla_{x_i} p^0 - \nabla_{y_i} p^1 + \mu_0 \Delta_y v_i^0 \quad \text{in } Y_{\mathrm{f}}, \tag{14a}$$

$$\rho^*\nabla_{y_i} v_i^0 = 0 \quad \text{in } Y_{\mathrm{f}}, \tag{14b}$$

$$v_i^0 = 0 \quad \text{at } \Gamma_{\mathrm{fs}}. \tag{14c}$$

N.B.: The microscopic problem (14) is analogous to ones defined by Giorgi's [7] and Chen et al. [3] for the nonlinear incompressible fluid flow except that here the gravity term $\rho^* g_i$ is in the $\varepsilon^{1/2}$ and not in the ε^0 term.

2.3. Variational formulation of the microscopic problem

1. In order to solve the microscopic boundary value problem (14) on the unit cell Y, we split the heterogeneous velocity v^h in two components, as suggested by Giorgi [7]

$$\mathbf{v}(\mathbf{x},\mathbf{y}) = \tilde{\mathbf{v}}(\mathbf{x},\mathbf{y}) + \langle\mathbf{v}\rangle(\mathbf{x}) \tag{15}$$

with $\langle \mathbf{v} \rangle(\mathbf{x}) = 1/|Y_\mathrm{f}| \int_{Y_\mathrm{f}} \mathbf{v}(\mathbf{x}, \mathbf{y})\,\mathrm{d}y$ being the average macroscopic velocity on Y_f. In the process of deriving a macroscopic equation for $\langle \mathbf{v^0} \rangle(\mathbf{x})$, the inertial nonlinear term of (14a) is linearized as follows:

$$\rho^*(\mathbf{x}) v_k^0(\mathbf{x}, \mathbf{y}) \cdot \nabla_{y_k} v_i^0(\mathbf{x}, \mathbf{y}) \cong \rho^*(\mathbf{x}) \langle v_k^0 \rangle(\mathbf{x}) \cdot \nabla_{y_k} v_i^0(\mathbf{x}, \mathbf{y}) \tag{16}$$

and the first term of the velocity expansion and the second term of the pressure one are split

$$\mathbf{v^0}(\mathbf{x}, \mathbf{y}) = \mathbf{a}(\mathbf{x}, \mathbf{y}) + \mathbf{b}(\mathbf{x}, \mathbf{y}, \langle \mathbf{v^0} \rangle), \tag{17}$$

$$\nabla_{y_i} p^1(\mathbf{x}, \mathbf{y}) = \nabla_{y_i} p_a^1(\mathbf{x}, \mathbf{y}) + \nabla_{y_i} p_b^1(\mathbf{x}, \mathbf{y}). \tag{18}$$

Introducing (17) and (18) in the local momentum (14a), we obtain for the velocity field $\mathbf{a}(\mathbf{x}, \mathbf{y})$

$$-\mu_0 \Delta_y a_i(\mathbf{x}, \mathbf{y}) = -[\nabla_{x_i} p^0(\mathbf{x}) + \nabla_{y_i} p_a^1(\mathbf{x}, \mathbf{y})] \tag{19}$$

and for the velocity $b(\mathbf{x}, \mathbf{y}, \langle \mathbf{v^0} \rangle)$

$$-\mu_0 \Delta_y b_i + \rho^* \langle v_k^0 \rangle(\mathbf{x}) \nabla_{y_k} b_i = -\nabla_{y_i} p_b^1(\mathbf{x}, \mathbf{y}) - \rho^* \langle v_k^0 \rangle(\mathbf{x}) \nabla_{y_k} a_i(\mathbf{x}, \mathbf{y}). \tag{20}$$

2. Eq. (19) is well studied, since Sanchez–Palencia [12] has shown that its variational formulation provides the macroscopic Darcy law. Note that this equation does not have a gravity term in its expression. In fact, Eq. (19) provides solution $\mathbf{a}(\mathbf{x}, \mathbf{y}) \in V_\mathrm{f}$, given by

$$a_i(\mathbf{x}, \mathbf{y}) = \frac{\omega_i^j(\mathbf{y})}{\mu_0}[-\nabla_{x_j} p^0(\mathbf{x})], \tag{21}$$

where:

- V_f is the space of periodic velocities defined by $V_\mathrm{f} = \{\nabla_{y_i} q_i = 0, q_i = 0 \text{ on } \Gamma_\mathrm{fs} \text{ and } q_i \text{ is } Y\text{-periodic}\}$
- each additional velocity field $\omega^j \in V_\mathrm{f}$, $j = 1, 2, 3$ must satisfy

$$\int_{Y_\mathrm{f}} \nabla_{y_k} \omega^j \cdot \nabla_{y_k} \mathbf{q}\,\mathrm{d}y = \int_{Y_\mathrm{f}} q_j(\mathbf{y})\,\mathrm{d}y \quad \forall \mathbf{q} \in V_\mathrm{f}. \tag{22}$$

Averaging solution (21) of $\mathbf{a}(\mathbf{x}, \mathbf{y})$ on the unit cell Y provides following Darcy law for the equivalent material

$$\mu_0 \langle a_i \rangle(\mathbf{x}) = P_{ij}[-\nabla_{x_j} p^0(\mathbf{x})] \tag{23}$$

with P_{ij} being the permeability tensor defined by

$$P_{ij} = \langle \omega_i^j \rangle = \frac{1}{|Y|} \int_Y \omega_i^j(\mathbf{y})\,\mathrm{d}y. \tag{24}$$

N.B.: The constant permeability tensor P_{ij} corresponds to the mean value of the component *i* of the velocity ω^j on the unit cell. This tensor is symmetric and positive definite. Its components depend only on the geometry of the unit cell Y and not on thermophysical data like viscosity or density.

3. After introducing solution (21) in Eq. (20) and taking into account that the variational expression of the periodic gradient $\nabla_{y_i} p_i^1$ has no mean contribution on the a unit cell [12], following variational formulation is used to solve Eq. (20) on the fluid part Y_f of the unit cell

$$\begin{aligned} &\mu_0 \int_{Y_\mathrm{f}} \nabla_{y_k} \mathbf{b} \cdot \nabla_{y_k} \mathbf{q}\,\mathrm{d}y + \rho^* \langle v_k^0 \rangle(\mathbf{x}) \int_{Y_\mathrm{f}} \nabla_{y_k} \mathbf{b} \cdot \mathbf{q}\,\mathrm{d}y \\ &\quad = -\frac{1}{\mu_0} \rho^* \langle v_k^0 \rangle [-\nabla_{x_j} p^0(\mathbf{x})] \int_{Y_\mathrm{f}} \nabla_{y_k} \boldsymbol{\omega}^\mathbf{j} \cdot \mathbf{q}\,\mathrm{d}y \quad \forall \mathbf{q} \in V_F. \end{aligned} \tag{25}$$

Per analogy to solution (21), there must also exists a solution for the microscopic periodic velocity field $\mathbf{b}(\mathbf{x}, \mathbf{y}, \langle\mathbf{v^0}\rangle)$ which satisfies the weak formulation (25)

$$\mathbf{b}(\mathbf{x}, \mathbf{y}, \langle\mathbf{v^0}\rangle) = \frac{1}{\mu_0}[-\nabla_{x_j} p^0(\mathbf{x})]\mathbf{h^j}(\mathbf{y}, \langle\mathbf{v^0}\rangle), \tag{26}$$

where the additional velocity fields $\mathbf{h}^j \in V_\mathrm{f}$, $j = 1, 2, 3$ are solutions of following special flow problems on the fluid part of the unit cell Y_f

$$\mu_0 \int_{Y_\mathrm{f}} \nabla_{y_k}\mathbf{h^j} \cdot \nabla_{y_k}\mathbf{q}\,\mathrm{d}y + \rho^*\langle v_k^0\rangle(\mathbf{x}) \int_{Y_\mathrm{f}} \nabla_{y_k}\mathbf{h^j} \cdot \mathbf{q}\,\mathrm{d}y = -\rho^*\langle v_k^0\rangle \int_{Y_\mathrm{f}} \nabla_{y_k}\boldsymbol{\omega}^\mathbf{j} \cdot \mathbf{q}\,\mathrm{d}y \quad \forall\mathbf{q} \in V_F. \tag{27}$$

4. Both velocity solutions $\mathbf{a}(\mathbf{x}, \mathbf{y})$ (21) and $\mathbf{b}(\mathbf{x}, \mathbf{y}, \langle\mathbf{v^0}\rangle)$ (26) are introduced in the splitting (17) of the microscopic periodic velocity field and then averaged over the whole unit cell Y. Proceeding that way, we obtain following generalized Forchheimer law:

$$\mu_0\langle v_i^0\rangle(\mathbf{x}) = [-\nabla_{x_j} p^0(\mathbf{x})][\langle\omega_i^j\rangle + \langle h_i^j\rangle(\langle\mathbf{v}^0\rangle)]. \tag{28}$$

This expression generalizes Darcy's law by adding a term which is nonlinear in the macroscopic velocity and leads to a mean velocity dependent permeability definition

$$P_{ij} = \langle\omega_i^j\rangle + \langle h_i^j\rangle(\langle\mathbf{v}^0\rangle). \tag{29}$$

3. Mixed finite element discretization of the unit cell problems

The numerical evaluation of the permeability tensor by the expression (29) needs in priority to determine the unknown periodic microscopic velocity fields $\boldsymbol{\omega}^\mathbf{j}$ and $\mathbf{h^j}$ with $j = 1, 2, 3$ by solving the weak forms (22) and (27) on the fluid part of unit cell. As both unknown velocity fields belong to the space V_f and thus are incompressible on the unit cell, the finite element discretization of the unit cell problems (22) and (27) becomes delicate.

3.1. Microscopic Stokes flow problems

1. Consider at first the constraint weak variational problems (22). Each of them corresponds to a *special Stokes flow problem* on the unit cell Y:

$$\varDelta_y\boldsymbol{\omega}^\mathbf{j} = \mathbf{e^j} \quad \text{on } Y_\mathrm{f} \text{ with } \mathbf{e^j} \text{ a unit vector and } \boldsymbol{\omega}^\mathbf{j} = 0 \text{ on } \varGamma_{\mathrm{fs}}, \tag{30}$$

$$\nabla_{y_i}\omega_i^j = 0 \quad \text{on } Y_\mathrm{f}. \tag{31}$$

This flow is characterized by a unit viscosity and driven either by a negative unit macro-pressure gradient or by a positive unit body force in direction j.

It is well-known that incompressible Stokes flow problems like (30) and (31) need to be stabilized. In the literature, several stabilization methods have been published: the *streamline-upwind-Petrov–Galerkin* (SUPG) technique [2], the *Galerkin least-square* (GLS) method ([6,13,4]) and, recently, the *variational multiscale* approach [8]. The last method is the most general and powerful one. But, as it needs two level of finite element discretization, it is also more complex to implement. Therefore, in this paper the GLS stabilization method has been adopted. Its key idea is to augment the Galerkin formulation with a weighted least-square form of the residuals of the corresponding Euler–Lagrange equation. Their weighting factors are stabilization parameters which are so designed that the method provides at least the exact solution for the 1-D case.

Let then the fluid domain Y_f of the unit cell be discretized by finite elements of size d_e. In addition to the periodic velocity weighting and trial solution spaces ($V_\mathbf{q}^d$ and $S_{\boldsymbol{\omega}^\mathbf{j}}^d$, respectively) a pressure space S_p^d is required. Because there are no explicit pressure boundary conditions, S_p^d suffices for both, weighting and trial solution. The GLS formulation

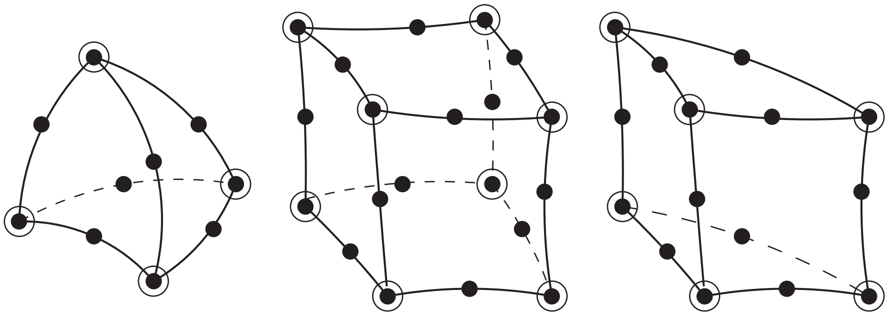

Fig. 2. Taylor–Hood volume elements (•: $\mathbf{q}^d$; ○: p^d).

of the jth special Stokes problem (30) corresponds to expression (22) plus:

- a weak expression of the continuity (31)

$$\int_{Y_\mathrm{f}} (p + m^d)[\nabla_k \cdot (\omega_k^j + q_k^d)]\,\mathrm{d}y \quad \forall m^d \in S_p^d, \tag{32}$$

  where the Lagrange multiplier is identified to the hydrostatic pressure p plus an arbitrary pressure weighting function m^d;
- a pressure stabilization term, named PSPG (pressure stabilizing/Petrov–Galerkin) [4]. It is defined by the sum of an element weighted residual of the Stokes equation (30)

$$ST_\tau = \sum_e \tau_e \int_{\Omega_e} \nabla(p + m^d) \cdot [\Delta_y(\boldsymbol{\omega}^\mathbf{j} + \mathbf{q}^d) - \mathbf{e}^\mathbf{j}]\,\mathrm{d}y. \tag{33}$$

Adding expressions (32)–(33) to the variational expression (22), we obtain following weak form of the constrained problem (30)–(31):

$$\begin{aligned} &\int_{Y_\mathrm{f}} \nabla_{y_k}\omega^j \cdot \nabla_{y_k}\mathbf{q}^d\,\mathrm{d}y + \sum_e \tau_e \int_{\Omega_e} \nabla(p + m^d) \cdot [\Delta_y(\boldsymbol{\omega}^\mathbf{j} + \mathbf{q}^d) - \mathbf{e}^\mathbf{j}]\,\mathrm{d}y \\ &\quad + \int_{Y_\mathrm{f}} (p + m^d) \cdot \nabla_{y_i}\omega_i^j\,\mathrm{d}y = \int_{Y_\mathrm{f}} q_j^d(\mathbf{y})\mathrm{d}y \quad \forall \mathbf{q}^d \in V_\mathbf{q}^d;\ \forall m^d \in S_p^d. \end{aligned} \tag{34}$$

Expressed in terms of bilinear forms, the weak form (34) becomes

$$\begin{aligned} &a(\nabla_y\boldsymbol{\omega}^\mathbf{j}, \nabla_y\mathbf{q}^d)_{Y_\mathrm{f}} + \sum_e (\tau_e\nabla(p + m^d), \Delta_y(\boldsymbol{\omega}^\mathbf{j} + \mathbf{q}^d) - \mathbf{e}^\mathbf{j})_{\Omega_e} \\ &\quad + (p + m^d, \nabla_y \cdot (\boldsymbol{\omega}^\mathbf{j} + \mathbf{q}^d))_{Y_\mathrm{f}} - (\mathbf{q}^d, \mathbf{e}^\mathbf{j})_{Y_\mathrm{f}} = 0. \end{aligned} \tag{35}$$

Thanks to the stabilization term ST_τ the restrictions of the LBB stability criteria [6] are circumvented and linear equal-order velocity–pressure interpolations can be used.

2. Classical mixed Q2Q1 and P2P1 Taylor–Hood volume elements (see Fig. 2) are used here to realized the space discretization. The velocity field $\boldsymbol{\omega}^\mathbf{j}$ and the weighting function $\mathbf{q}^h$ are approximated by quadratic shape functions:

$$\boldsymbol{\omega}^\mathbf{j} = \sum_\alpha N_\alpha W_\alpha^j = \mathbf{N}\mathbf{W}^\mathbf{j}; \quad \mathbf{q}^d = \sum_\alpha N_\alpha Q_\alpha = \mathbf{N}\mathbf{Q}. \tag{36}$$

Whereas, the pressure field p and its weighting function m^d are approximated by linear shape functions:

$$p = \sum_\beta \tilde{N}_\beta P_\beta = \tilde{\mathbf{N}}\mathbf{P}, \tag{37a}$$

$$m^d = \sum_\beta \tilde{N}_\beta M_\beta = \tilde{\mathbf{N}}\mathbf{M}. \tag{37b}$$

At next, the stationary of the weak form (34) is expressed by using the discretized velocity and pressure fields (36)–(37). This stationarity condition leads to the global matrix equation, which is linear and can be written in the following partitioned form by segregating nodal velocity values from pressure ones

$$\begin{pmatrix} \bar{\mathbf{K}} & \mathbf{G}+\mathbf{D}(\tau) \\ \mathbf{G}^{\mathrm{T}}+\mathbf{D}^{\mathrm{T}}(\tau) & \mathbf{0} \end{pmatrix} \begin{Bmatrix} \mathbf{W}^j \\ \mathbf{P} \end{Bmatrix} = \begin{Bmatrix} \mathbf{F}^j \\ \mathbf{L}^j(\tau) \end{Bmatrix} \tag{38}$$

with $\bar{\mathbf{K}} = \int_{Y_\mathrm{f}} (\nabla\mathbf{N})^{\mathrm{T}} \cdot \nabla\mathbf{N}\, \mathrm{d}y$: symmetric, positive-definite stiffness matrix; $\mathbf{G}^{\mathrm{T}} = \int_{Y_\mathrm{f}} \tilde{\mathbf{N}}^{\mathrm{T}} \cdot (\nabla\cdot\mathbf{N})\, \mathrm{d}y$: discrete divergence operator; $\mathbf{D}(\boldsymbol{\tau})^{\mathrm{T}} = \sum_e \tau_e \int_{\Omega_e} (\nabla\tilde{\mathbf{N}})^{\mathrm{T}} \cdot (\nabla\cdot\nabla\mathbf{N})\, \mathrm{d}y$: discrete stabilization Laplace term; $\mathbf{F}^j = \int_{Y_\mathrm{f}} \mathbf{N}^{\mathrm{T}} \cdot \mathbf{e}^j\, \mathrm{d}y$: discrete body force term in direction j; $\mathbf{L}^j(\tau) = \sum_e \tau_e \int_{\Omega_e} (\nabla\tilde{\mathbf{N}})^{\mathrm{T}} \cdot \mathbf{e}^j\, \mathrm{d}y$: discrete stabilization right-hand term.

3. The numerical results in [10] of the special Stokes flow problems (22) on the unit cell Y illustrate that the velocity fields ω^j are low ($\approx 3\,\mu\mathrm{m/s}$) and the corresponding Reynolds numbers are small. Therefore, following the suggestion of Tezduyar et al. [13], we adopt their low-Reynolds limit as local stabilization factor

$$\tau_e = \frac{d_e^2}{12} \tag{39}$$

with d_e being the element length, defined so that it is equal to the diameter of a sphere which is volume-equivalent to the considered element.

3.2. Microscopic additional flow problems

1. As the unknown velocity fields $\mathbf{h^j}$ with $j = 1, 2, 3$ belong also to the divergence-free space V_f, the discretization of the additional flow problems (27) is realized in a similar way to the special Stokes flow problems. Again a weak expression of the continuity (31) and a weighted residual of the additional flow equations (27) are added as pressure stabilization to the variational equations (27) discretized by mixed finite elements. Indeed, we have again that: $\mathbf{h^j} \in S^d_{\mathbf{h}^j}$ and $\mathbf{q^d} \in V^d_{\mathbf{q}}$. Whereas, the pressure field p and its weighting function m^d belong to S^d_p. Note that also the same spatial discretization as for the Stokes flow problem is used here: mixed Q2Q1 and P2P1 Taylor–Hood volume elements.

Using the definitions of bilinear forms, the modified Galerkin form is given by

$$\mu_0 a(\nabla_y \mathbf{h^j}, \nabla_y \mathbf{q}^d)_{Y_\mathrm{f}} + (p+m^d, \nabla_y\cdot(\mathbf{h^j}+\mathbf{q}^d))_{Y_\mathrm{f}} + \sum_e (\tau_e \nabla(p+m^d), \mathbf{R^d_e})_{\Omega_e} + (\rho^*\langle\mathbf{v^0}\rangle\cdot\nabla_y\mathbf{h^j}, \mathbf{q}^d)_{Y_\mathrm{f}}$$
$$= -(\rho^*\langle\mathbf{v^0}\rangle\cdot\nabla_y\boldsymbol{\omega}^\mathbf{j}, \mathbf{q}^d)_{Y_\mathrm{f}} \tag{40}$$

with R^d_e, the element residual of the momentum equation given by

$$\mathbf{R^d_e} = \mu_0 \Delta_y(\mathbf{h^j}+\mathbf{q}^d) + \rho^*\langle\mathbf{v^0}\rangle\cdot\nabla_y(\mathbf{h^j}+\mathbf{q}^d) + \rho^*\langle\mathbf{v^0}\rangle\cdot\nabla_y\boldsymbol{\omega}^\mathbf{j}. \tag{41}$$

The weak form (40) presents an advective–diffusive character. Indeed, the fourth bilinear term is responsible for advection of the velocity field $\mathbf{h^j}$. The right-hand side term is known because the Stokes velocities $\boldsymbol{\omega}^\mathbf{j}$ have been previously determined by solving the matrix equation system (38). Note that, if a linked macroscopic 3-D conjugate heat and fluid flow analysis [1] have been previously performed, the corresponding mean velocity $\langle\mathbf{v^0}\rangle(\mathbf{x})$ is preferred to the one obtained by solving the nonlinear Forchheimer law (28) for the equivalent homogeneous material.

In order to stabilize also advection and complete thus the GLS formulation for the additional flow problems (27), a SUPG stabilization term, named ST_γ and defined by

$$ST_\gamma = \sum_e (\gamma_e \nabla_y\cdot\mathbf{q}^d, \nabla_y\cdot\mathbf{h^j})_{\Omega_e}, \tag{42}$$

where γ_e is a numerical bulk viscosity [4], is also added to the weak form (40)

$$\mu_0 a(\nabla_y \mathbf{h}^{\mathbf{j}}, \nabla_y \mathbf{q}^d)_{Y_\mathrm{f}} + (p + m^d, \nabla_y \cdot (\mathbf{h}^{\mathbf{j}} + \mathbf{q}^d))_{Y_\mathrm{f}} + \sum_e \tau_e (\nabla(p + m^d), \mathbf{R}_e^d)_{\Omega_e}$$
$$+ (\rho^* \langle \mathbf{v}^{\mathbf{0}} \rangle \cdot \nabla_y \mathbf{h}^{\mathbf{j}}, \mathbf{q}^d)_{Y_\mathrm{f}} + \sum_e \gamma_e (\nabla_y \cdot \mathbf{q}^d, \nabla_y \cdot \mathbf{h}^{\mathbf{j}})_{\Omega_e} = -(\rho^* \langle \mathbf{v}^{\mathbf{0}} \rangle \cdot \nabla_y \boldsymbol{\omega}^{\mathbf{j}}, \mathbf{q}^d)_{Y_\mathrm{f}} \tag{43}$$

2. At next, the stationarity condition of the weak form (43) is formulated and the spatial discretization of the additional velocities: $\mathbf{h}^{\mathbf{j}} = \sum_\alpha N_\alpha H_\alpha^j = \mathbf{N}\mathbf{H}^{\mathbf{j}}$, their weighting functions (36), the pressure field (37a) and their associated weighting functions (37b) are introduced in its expression. This condition leads again to a global linear matrix system to solve numerically. Its expression is given by:

$$\begin{pmatrix} \mathbf{K} + \mathbf{A} + \mathbf{S}(\gamma) & \mathbf{G} + \mathbf{D}(\tau) + \mathbf{B}(\tau) \\ \mathbf{G}^\mathrm{T} + \mathbf{D}^\mathrm{T}(\tau) + \mathbf{B}^\mathrm{T}(\tau) & \mathbf{0} \end{pmatrix} \begin{Bmatrix} \mathbf{H}^j \\ \mathbf{P} \end{Bmatrix} = \begin{Bmatrix} -\mathbf{J} \cdot \mathbf{W}^j \\ -\mathbf{B}(\tau) \cdot \mathbf{W}^j \end{Bmatrix} \tag{44}$$

with $\mathbf{K} = \int_{Y_\mathrm{f}} (\nabla \mathbf{N})^\mathrm{T} \cdot \mathbf{C} \cdot \nabla \mathbf{N} \, \mathrm{d}y$: viscous stiffness matrix with $\mathbf{C} = \mu_0 \mathbf{I}$; $\mathbf{A} = \int_{Y_\mathrm{f}} \mathbf{N}^\mathrm{T} \cdot (\rho^* \langle \mathbf{v}^{\mathbf{0}} \rangle \cdot \nabla \mathbf{N}) \, \mathrm{d}y$: advective stiffness contribution; $\mathbf{S}(\gamma) = \sum_e \gamma_e \int_{\Omega_e} (\nabla \cdot \mathbf{N})^\mathrm{T} \cdot (\nabla \cdot \mathbf{N}) \, \mathrm{d}y$: SUPG stiffness contribution; $\mathbf{G} = \int_{Y_\mathrm{f}} (\nabla \cdot \mathbf{N})^\mathrm{T} \cdot \tilde{\mathbf{N}} \, \mathrm{d}y$: transpose of the discrete divergence operator; $\mathbf{D}(\boldsymbol{\tau})^\mathrm{T} = \sum_e \tau_e \int_{\Omega_e} (\nabla \tilde{\mathbf{N}})^\mathrm{T} \cdot \mathbf{C} \cdot (\nabla \cdot \nabla \mathbf{N}) \, \mathrm{d}y$: discrete viscous stabilization term; $\mathbf{B}(\boldsymbol{\tau}) = \sum_e \tau_e \int_{\Omega_e} (\rho^* \langle \mathbf{v}^{\mathbf{0}} \rangle \cdot \nabla \mathbf{N})^\mathrm{T} \cdot \nabla \tilde{\mathbf{N}} \, \mathrm{d}y$: advective term due to PSPG stabilization; $\mathbf{J} = \int_{Y_\mathrm{f}} \mathbf{N}^\mathrm{T} \cdot (\rho^* \langle \mathbf{v}^{\mathbf{0}} \rangle \cdot \nabla \mathbf{N}) \, \mathrm{d}y$: discrete RHS matrix; $\mathbf{W}^j$: solution of flow problem (38) for unit body forces in j direction.

3. As the mean velocity $\langle \mathbf{v}^{\mathbf{0}} \rangle(\mathbf{x})$ in the cooling channels of turbomachines components may be large, it is more suitable to adopt stabilization parameters which are function of the Reynolds number, like proposed by Franca and Hughes [6]

$$\gamma_e = \frac{d_e \|\mathbf{h}_\mathbf{e}^{\mathbf{j}}\|}{\zeta(Re_e)}; \quad \tau_e = \frac{d_e \zeta(Re_e)}{2\|\mathbf{h}_\mathbf{e}^{\mathbf{j}}\|} \tag{45}$$

with

- $\zeta(Re_e) = \min\,(Re_e/3, 1)$ with $Re_e = \rho^* \|\mathbf{h}_\mathbf{e}^{\mathbf{j}}\| d_e / 2\mu_0$: the local Reynolds number;
- $\|\mathbf{h}_e^j\|$: the L_2 norm of the local element velocity.

4. Conclusions

A multiscale approach based on the homogenization method has been applied to the compressible gas flow through a periodic porous material in turbomachines. This formulation allows to deduce as macroscopic behavior a generalized Forchheimer law with a permeability function of the mean velocity. Significant nonlinear inertial effects at the microscale induces these more general equivalent law. To express this equivalent permeability, two kinds of additional microscopic flow problems have been specified on the unit cell. As the microscopic velocity fields must be divergence-free, a mixed stabilized finite element discretization is required. The GLS formalism is used here to realize stabilization in a weak sense. This method leads to specify a pressure stabilization term for the microscopic Stokes flow problems and a further SUPG stabilization term for the additional flow problems. The expression of the stationarity of both discretized weak forms allows to establish explicitly the two linear matrix systems to solve on the unit cell. Their implementation in the homogenization program HOMAT [9] is, under progress. After its validation, nonlinear Forchheimer corrections to the effective Darcy permeabilities will be evaluated as soon as possible for curved and flat transpiration cooled multilayer plates [10].

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